

# Mechanics: Oscillations (Solutions)

FIZIKA SPhO Training

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## Contents

<b>1</b>	<b>Notes</b>	<b>2</b>
1.1	Simple Harmonic Motion . . . . .	2
1.1.1	Characteristic Equation . . . . .	2
1.1.2	Physical Problems . . . . .	3
1.1.3	Small Oscillations . . . . .	5
1.2	Energy in Oscillatory Systems . . . . .	6
1.3	Damped/Driven Harmonic Oscillators . . . . .	7
1.3.1	Solving the Damped Harmonic Oscillator . . . . .	8
1.3.2	Case 1: Underdamped Oscillations . . . . .	8
1.3.3	Case 2: Critically Damped Oscillations . . . . .	9
1.3.4	Case 3: Overdamped Oscillations . . . . .	9
1.3.5	Solving the Driven Harmonic Oscillator . . . . .	10
1.3.6	Resonance . . . . .	10
<b>2</b>	<b>Problems</b>	<b>12</b>
<b>3</b>	<b>Advanced Problems</b>	<b>25</b>

# 1 Notes

## 1.1 Simple Harmonic Motion

### 1.1.1 Characteristic Equation

Simple harmonic motion (SHM) is the motion of a particle in which

1. its acceleration is directly proportional to its displacement from a fixed equilibrium position
2. it is always directed towards that equilibrium position.

Equivalently, if  $x(t)$  denotes the displacement at time  $t$ , the defining equation of motion is:

$$a = \frac{d^2x}{dt^2} = -\omega^2 x \quad (1)$$

where the angular frequency  $\omega = 2\pi f = \frac{2\pi}{T}$ .

By solving the differential equation, the general equation of motion is

$$x(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t) \quad (2)$$

for some constants  $C_1$  and  $C_2$ , depending on the initial conditions.

Equivalently, using the R-formula,

$$x(t) = x_0 \cos(\omega t + \varphi) \quad (3)$$

where  $x_0$  is the amplitude, and  $\varphi$  is the phase.

There are other useful equations that stem from the characteristic equation:

- Maximum Values:  $v_{\max} = A\omega$ ,  $a_{\max} = A\omega^2$
- Total Energy:  $E = \frac{1}{2}kA^2 = \frac{1}{2}m\omega^2A^2$
- Kinetic Energy:  $K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2(A^2 - x^2)$
- Potential Energy:  $U = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2x^2$

**Example 1.1.** Try to derive the functions  $v(x)$  and  $a(v)$  and then plot them.

Eliminate the time variable with  $\sin^2\theta + \cos^2\theta = 1$ :

$$v^2 = A^2\omega^2(1 - \cos^2\theta) = \omega^2(A^2 - x^2) \implies \boxed{v(x) = \pm\omega\sqrt{A^2 - x^2}}.$$

Use the change-of-variables trick  $a = v dv/dx$ . From  $v^2 = \omega^2(A^2 - x^2)$ :

$$\frac{dv}{dx} = \frac{-\omega^2 x}{v} \implies a = v \frac{dv}{dx} = -\omega^2 x.$$

Solving the earlier relation for  $x$  in terms of  $v$ ,  $x = \pm\sqrt{A^2 - \frac{v^2}{\omega^2}}$ , gives the *multi-valued* (upper + or lower - branch) expression

$$\boxed{a(v) = \mp\omega^2\sqrt{A^2 - \frac{v^2}{\omega^2}}}.$$

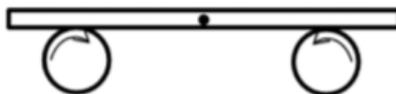
Features of Graphs:

- $a(x)$  **straight line:** slope  $-\omega^2$ ; endpoints  $(x = \pm A, a = \mp A\omega^2)$ .
- $v(x)$  **ellipse:** intercepts at  $(x = \pm A, v = 0)$  and  $(x = 0, v = \pm A\omega)$ .
- $a(v)$  **ellipse:** intercepts at  $(v = \pm A\omega, a = 0)$  and  $(v = 0, a = \pm A\omega^2)$ .

### 1.1.2 Physical Problems

To solve such problems, we should always aim to write the equations of motion using **N2L**.

**Example 1.2.** A rod of mass  $m$  is put on top of two oppositely rotating cylinders with coefficient of friction  $\mu$ , separated by a distance  $2L$ . The centers of the cylinders are fixed. If the rod is displaced from equilibrium, describe the subsequent motion of the rod, and calculate a property of the motion.



Denote the normal reactions at the left and right supports by  $N_1$  and  $N_2$ , respectively. To begin, suppose the rod was displaced by a distance  $x$  from equilibrium to right:

$$\begin{aligned} N_1 + N_2 &= mg \quad (\text{vertical force balance}), \\ N_1(L + x) &= N_2(L - x) \quad (\text{torque balance about the centre of mass}). \end{aligned}$$

Solving,

$$N_2 = \frac{mg(L + x)}{2L}, \quad N_1 = \frac{mg(L - x)}{2L}.$$

The magnitude of the kinetic-friction force at each support is  $\mu N_i$ .

$$F_{f,1} = +\mu N_1 \quad (\text{to the right}), \quad F_{f,2} = -\mu N_2 \quad (\text{to the left}).$$

Hence the net horizontal force on the rod is

$$F = \mu N_1 - \mu N_2 = \mu mg \left( \frac{L - x}{2L} - \frac{L + x}{2L} \right) = -\frac{\mu mg}{L} x.$$

$$m\ddot{x} = -\frac{\mu mg}{L} x \quad \implies \quad \ddot{x} + \underbrace{\left( \frac{\mu g}{L} \right)}_{\omega^2} x = 0.$$

This is the characteristic differential equation of *simple harmonic motion* with angular frequency

$$\omega = \sqrt{\frac{\mu g}{L}}$$

**Example 1.3.** To tackle problems where the SHM takes place about a shifted equilibrium, we adopt a general method, illustrated with the following example: A mass  $m$  hangs from a vertical massless spring of stiffness  $k$  with zero natural length. Let  $y$  be the height from the ceiling to the mass attached to the spring.

1. **Find the static equilibrium:**

$$k y_{\text{eq}} = m g \quad \implies \quad y_{\text{eq}} = \frac{m g}{k}.$$

2. Define a new variable, displacement about equilibrium:

$$\delta y(t) = y(t) - y_{eq},$$

so that  $\delta y = 0$  at the static equilibrium.

3. Write the new equation of motion:

$$m\ddot{y} = mg - ky \quad \Rightarrow \quad m(\ddot{\delta y} + \ddot{y}_{eq}) = mg - k(\delta y + y_{eq})$$

Since  $y_{eq}$  is constant,  $\ddot{y}_{eq} = 0$ . Substituting in  $y_{eq}$ , we get

$$m\ddot{\delta y} + k\delta y = 0.$$

4. Apply SHM solution:

$$\ddot{\delta y} + \omega^2 \delta y = 0, \quad \omega = \sqrt{\frac{k}{m}},$$

It turns out the frequency stays the same.

**Example 1.4.** A solid cube of side length  $b$  with density  $\rho_c$  is floating in a certain liquid with density  $\rho_l$ . It is pushed down slightly then released. You may assume the liquid level stays fixed (there is no displacement of liquid). Describe the subsequent motion in detail.

Let's follow the same framework in the previous example. Let  $h_0$  be the submerged depth of the cube at rest. Buoyant force = weight:

$$\rho_l g (b^2 h_0) = \rho_c g (b^3) \quad \Rightarrow \quad h_0 = \frac{\rho_c}{\rho_l} b.$$

Define a small vertical displacement

$$\delta y(t) = h(t) - h_0,$$

measured *downward*. Thus  $\delta y > 0$  means the cube is pushed farther into the liquid, and  $\delta y = 0$  corresponds to equilibrium.

The submerged volume becomes  $V_{\text{sub}} = b^2(h_0 + \delta y)$ ; hence the buoyant force is

$$F_B = \rho_l g b^2 (h_0 + \delta y).$$

The cube's weight is unchanged,  $mg = \rho_c g b^3$ . The net upward force is

$$F_{\text{net}} = F_B - mg = \rho_l g b^2 (h_0 + \delta y) - \rho_c g b^3.$$

Using the equilibrium relation  $\rho_l g b^2 h_0 = \rho_c g b^3$  cancels the constants:

$$F_{\text{net}} = -\rho_l g b^2 \delta y.$$

Newton's 2nd Law with mass  $m = \rho_c b^3$ :

$$\rho_c b^3 \ddot{\delta y} = -\rho_l g b^2 \delta y \quad \Rightarrow \quad \ddot{\delta y} + \frac{\rho_l g}{\rho_c b} \delta y = 0.$$

The standard SHM form  $\ddot{\delta y} + \omega^2 \delta y = 0$  identifies

$$\omega = \sqrt{\frac{\rho_l g}{\rho_c b}}$$

Hence the cube oscillates vertically about its equilibrium depth with angular frequency  $\omega$ , provided  $\rho_c < \rho_l$  (so the cube floats).

### 1.1.3 Small Oscillations

A common trick very often used to tackle small oscillations is to use the binomial expansion and take low order terms:

$$(1+x)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k = 1 + nx + \frac{n(n-1)}{2!} x^2 + \dots$$

For  $|x| \ll 1$ , higher-order terms become negligible, so to first order

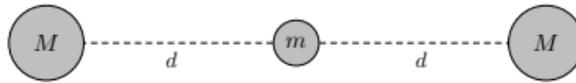
$$(1+x)^n \approx 1 + nx.$$

By taking first order terms, we essentially "linearize" many forces, hence many situations become simple harmonic. However, if your answer does not make physical sense, you might have to consider a 2nd order or higher term.

**Example 1.5** (SPOT TST 2022). Suppose that two point masses  $m_1$  and  $m_2$  separated by a distance  $r$  interact via the potential

$$U = \frac{\kappa m_1 m_2}{r^n}, \quad \kappa > 0, \quad n \in \mathbb{Z}_{>0}.$$

State, with brief reasons, whether the corresponding force is attractive or repulsive.



Now, two identical fixed masses  $M$  are a distance  $2d$  apart on a straight line. A third mass  $m$  is free to move along the line segment between them (so the separations are  $d \pm x$  with  $|x| \ll d$ ). Determine, in terms of the symbols introduced, the angular frequency  $\omega$  of the small oscillations of  $m$  about its equilibrium position.

To solve the first part of the question, note that the force corresponding to the given potential is

$$F(r) = -\frac{dU}{dr} = -\frac{d}{dr} \left( \frac{\kappa m_1 m_2}{r^n} \right) = +\frac{n\kappa m_1 m_2}{r^{n+1}} > 0,$$

directed *outwards* along the line joining the masses (it increases the separation). Hence the interaction is **repulsive**.

When the mobile mass  $m$  is displaced a small distance  $x$  to the right of the midpoint,

$$U(x) = \frac{\kappa m M}{(d+x)^n} + \frac{\kappa m M}{(d-x)^n}.$$

Factor out  $d^{-n}$  and expand each term with the binomial series:

$$(d \pm x)^{-n} = d^{-n} \left( 1 \pm \frac{x}{d} \right)^{-n} \simeq d^{-n} \left[ 1 \mp n \frac{x}{d} + \frac{1}{2} n(n+1) \left( \frac{x}{d} \right)^2 \right].$$

Adding the two expansions cancels the linear term, giving

$$U(x) = 2\kappa m M d^{-n} \left[ 1 + \frac{1}{2} n(n+1) \left( \frac{x}{d} \right)^2 \right] = U_0 + \kappa m M \frac{n(n+1)}{d^{n+2}} x^2.$$

Comparing  $U(x) = U_0 + \frac{1}{2}k_{\text{eff}}x^2$  with the last line,

$$k_{\text{eff}} = 2\kappa M n(n+1) d^{-(n+2)}.$$

For a particle of mass  $m$  in a quadratic potential,  $\frac{1}{2}m\omega^2 = \frac{1}{2}k_{\text{eff}}$ . Hence

$$\omega = \sqrt{\frac{k_{\text{eff}}}{m}} = \sqrt{\frac{2\kappa M n(n+1)}{d^{n+2}}}.$$

The motion is therefore simple harmonic about the midpoint provided the displacement is small compared with  $d$ . We notice here that we had to make a 2nd order expansion - had we expanded to 1st order, the answer would not have made physical sense. Try it yourself!

## 1.2 Energy in Oscillatory Systems

Any conservative mechanical system can be described by a potential energy function  $U(q)$ , where  $q$  is the generalized coordinate. Equilibrium positions  $q_0$  satisfy

$$\left. \frac{dU}{dq} \right|_{q=q_0} = 0.$$

At these points, there is no net force, since  $F = -\frac{dU}{dq}$

Additionally, note that even though an object is in equilibrium, it may be in unstable or stable equilibrium. This can be deduced from the second derivative of the potential energy. If  $U''(q) < 0$ , this indicates an unstable equilibrium. If  $U''(q) > 0$ , it indicates a stable equilibrium.

### Taylor Expansion around Equilibrium

For small displacements  $x = q - q_0$  about the stable equilibrium, expand  $U(q)$  in a Taylor series:

$$U(q) = U(q_0) + \left. \frac{dU}{dq} \right|_{q_0} x + \frac{1}{2} \left. \frac{d^2U}{dq^2} \right|_{q_0} x^2 + \dots$$

Since the first derivative at equilibrium vanishes as it is equal to 0 at equilibrium, to second order:

$$U(q) \approx U(q_0) + \frac{1}{2}U''(q_0)x^2.$$

We may drop the constant  $U(q_0)$  as it does not affect the dynamics. This was explained previously on page 6 (b).

1. Potential:  $U(x) = \frac{1}{2}U''(q_0)x^2, \quad x = q - q_0.$

2. Force from potential:  $F(x) = -\frac{dU}{dx} = -U''(q_0)x.$

3. Newton's 2nd law:  $m\ddot{x} = F(x) = -U''(q_0)x \implies m\ddot{x} + U''(q_0)x = 0.$

4. Compare with SHM:  $\ddot{x} + \omega^2 x = 0 \implies \omega^2 = \frac{U''(q_0)}{m}, \quad \boxed{\omega = \sqrt{\frac{U''(q_0)}{m}}}.$

**Example 1.6.** A particle of mass  $m$  moves in one dimension under the potential

$$U(x) = U_0 \left( \frac{x^2}{a^2} - 1 \right)^2,$$

where  $U_0 > 0$  and  $a > 0$  are constants.

- (a) Find all equilibrium points  $x_0$  by solving  $U'(x_0) = 0$ .
- (b) Determine which of these are stable equilibria.
- (c) Using a Taylor expansion of  $U(x)$  about the stable equilibrium  $x_0$ , show that for small oscillations the motion is simple harmonic with angular frequency

$$\omega = \sqrt{\frac{U''(x_0)}{m}}.$$

- (d) Compute  $U''(x_0)$  at the stable point and hence write down the explicit expression for  $\omega$  in terms of  $U_0$ ,  $m$  and  $a$ .

This example is a direct application of the formulas above.

- (a) We have

$$U(x) = U_0 \left( \frac{x^2}{a^2} - 1 \right)^2 = U_0 \left( \frac{x^4}{a^4} - 2 \frac{x^2}{a^2} + 1 \right).$$

Compute

$$U'(x) = U_0 \left( \frac{4x^3}{a^4} - \frac{4x}{a^2} \right) = \frac{4U_0 x}{a^2} \left( \frac{x^2}{a^2} - 1 \right).$$

Setting  $U'(x) = 0$  gives

$$x = 0, \quad \text{or} \quad \frac{x^2}{a^2} = 1 \implies x = \pm a.$$

- (b) Next,

$$U''(x) = U_0 \left( \frac{12x^2}{a^4} - \frac{4}{a^2} \right).$$

Evaluate at each equilibrium:

$$U''(0) = -\frac{4U_0}{a^2} < 0 \quad (\text{unstable}), \quad U''(\pm a) = U_0 \left( \frac{12a^2}{a^4} - \frac{4}{a^2} \right) = \frac{8U_0}{a^2} > 0 \quad (\text{stable}).$$

- (c) Around a *stable* equilibrium  $x_0 = \pm a$ , we write

$$U(x) \approx U(x_0) + \frac{1}{2} U''(x_0) (x - x_0)^2.$$

The effective potential near  $x_0$  is that of a spring,  $\frac{1}{2} m \omega^2 (x - x_0)^2$ , so

$$m \omega^2 = U''(x_0) \implies \omega = \sqrt{\frac{U''(x_0)}{m}}.$$

- (d) Since  $U''(x_0) = 8U_0/a^2$ , we get

$$\omega = \sqrt{\frac{8U_0}{m a^2}}.$$

### 1.3 Damped/Driven Harmonic Oscillators

In reality, many systems are not ideal and are subject to external forces. Forces such as friction and air resistance that "dissipate" damping lead to **damped oscillations**. On the other hand, forces that "force" an oscillation lead to **driven oscillations**.

### 1.3.1 Solving the Damped Harmonic Oscillator

To solve for the equation of motion, we write N2L with the damping force as the net force.

**Example 1.7.** Consider a typical horizontal mass-spring system with mass  $m$  and spring constant  $k$ , but subject to a damping drag force  $F_{drag} = -bv$  from air resistance, where  $b > 0$ . Find the general equation of motion.

By N2L,

$$m\ddot{x} = -kx - b\dot{x}$$

keeping in mind that the signs for both forces are negative, since both forces decrease the acceleration of the mass.

We may rearrange to get a 2nd order ODE:

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0$$

which can be solved using an exponential guess,  $x(t) = Ae^{st}$ . (Try it yourself!)

You should obtain

$$s = -\frac{b}{2m} \pm \sqrt{\frac{b^2}{4m^2} - \frac{k}{m}}$$

and thus, the general equation of motion is

$$x(t) = A_1e^{s_1t} + A_2e^{s_2t}$$

where  $s_1$  and  $s_2$  are the two solutions for  $s$ , and  $A_1$  and  $A_2$  are constants depending on the initial conditions.

We usually define the **damping coefficient**,  $\gamma$  and the **resonant/natural frequency**,  $\omega_0$ :

$$\gamma := \frac{b}{2m} > 0 \tag{4}$$

$$\omega_0 := \sqrt{\frac{k}{m}} > 0 \tag{5}$$

and thus,  $s$  is usually written as

$$s = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2} \tag{6}$$

There are **three possible cases** of damping, depending on the term inside the square root.

### 1.3.2 Case 1: Underdamped Oscillations

Underdamped oscillations occur when  $\gamma < \omega_0$ . This means  $s$  is imaginary. In this case, oscillations still exist. We can define the angular frequency as

$$\omega := \sqrt{\omega_0^2 - \gamma^2} = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \tag{7}$$

Thus,  $s = -\gamma \pm i\omega$ , so we get

$$x(t) = e^{-\gamma t} (A_1 \cos(\omega t) + A_2 \sin(\omega t)) \tag{8}$$

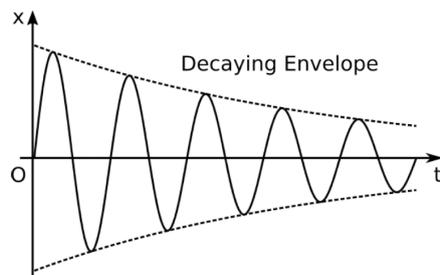


Figure 1: The term in front  $e^{-\gamma t}$  is the equation of the decaying envelope. As  $t \rightarrow \infty$ , the amplitude rightfully decays to zero as all energy is lost.

### 1.3.3 Case 2: Critically Damped Oscillations

Critically damped oscillations occur when  $\gamma = \omega_0$ . This means that  $s$  is repeated.

In this case, oscillations don't exist. In particular, this is when the system returns to equilibrium in the shortest time possible. As  $s = -\gamma, -\gamma$ , we have

$$x(t) = (A_1 + A_2 t) e^{-\gamma t} \quad (9)$$

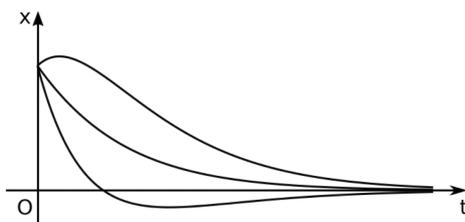


Figure 2: Critically damped systems decay to zero in the shortest time possible.

### 1.3.4 Case 3: Overdamped Oscillations

Overdamped oscillations occur when  $\gamma > \omega_0$ . This means that  $s$  is real and distinct.

In this case, oscillations don't exist. We simply have the general form

$$x(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \quad (10)$$

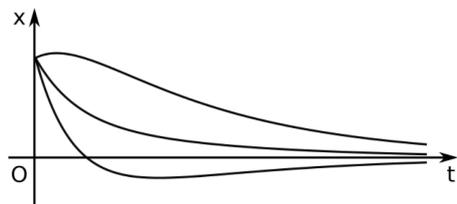


Figure 3: Overdamped systems do not have oscillations and decay to zero.

**Example 1.8.** Without referring to the steps above, try deriving the equations of motion for the 3 different cases yourself.

### 1.3.5 Solving the Driven Harmonic Oscillator

Again, we write N2L, accounting for the driving force. The driving force is usually sinusoidal in nature.

**Example 1.9.** Consider the situation in Example 1.5, but there is now an additional applied force  $F(t) = F_0 \cos(\omega t)$ . Find the general equation of motion.

By N2L,

$$F_0 \cos(\omega t) - b\dot{x} - kx = m\ddot{x} \quad \Rightarrow \quad \ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m} \cos(\omega t)$$

To solve this non-homogeneous ODE, you would need to consider  $x_{homogeneous}$  and  $x_{particular}$ , where  $x_{homogeneous}$  is given by either Equation (8), (9) or (10) depending on the case of damping.

The sinusoidal form of the RHS means that we should guess

$$x_{particular} = A \cos(\omega t) + B \sin(\omega t)$$

By differentiating twice and substituting into the above equation, you should eventually obtain

$$A = \frac{\frac{F_0}{m} (\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}, \quad B = \frac{2\frac{F_0}{m}\gamma\omega}{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}$$

The amplitude of the oscillation, by the R-formula, is thus given by

$$x_0 = \sqrt{A^2 + B^2} = \frac{\frac{F_0}{m}}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}}$$

Thus, the general equation of motion is

$$x(t) = x_{homogeneous} + x_{particular} = A_1 e^{s_1 t} + A_2 e^{s_2 t} + \frac{\frac{F_0}{m}}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}} \cos(\omega t + \varphi)$$

where all three cases of damping are accounted for by the values of  $s$ .

However, all three cases have something in common! They all **decay to zero eventually**. As such, we are usually not interested in the homogeneous solution, and we **only care about the particular solution**. The particular solution is also called the **steady-state solution**.

### 1.3.6 Resonance

Resonance is the state whereby there is **maximum energy transfer** from the driving force to the oscillating system.

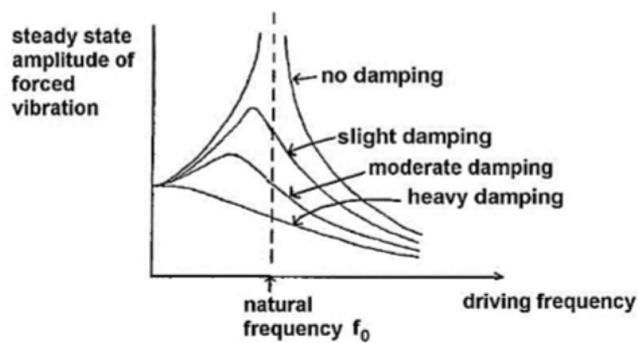
When there is **no damping**, resonance occurs when  $\omega = \omega_0$  (i.e. the driving frequency is equal to the natural frequency).

When there is **damping**, resonance occurs when the **amplitude is maximum**. From the form of  $x_0$ , we wish to minimise the term in the square root in the denominator,  $(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2$ , in order to maximise  $x_0$ . Thus,

$$\frac{d}{d\omega} \left( (\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2 \right) = 0 \quad \Rightarrow \quad \omega_{res} = \sqrt{\omega_0^2 - 2\gamma^2} \quad (11)$$

At the resonant frequency  $\omega_{res}$ , the maximum amplitude is

$$x_{0,max} = \frac{\frac{F_0}{m}}{2\gamma\sqrt{\omega_0^2 - \omega^2}} \quad (12)$$



## 2 Problems

*Problems are arranged in roughly increasing difficulty.*

**Problem 2.1** (SPhO 2018). (i) A horizontal platform vibrates with simple harmonic motion in the horizontal direction with a period of 2.0 s. A small object placed on the platform begins to slide when the amplitude of vibration reaches 0.4 m. Calculate the coefficient of static friction between the object and the platform. (ii) The platform now executes vertical simple harmonic motion with a period of 1.5 s. What is the maximum amplitude of the motion if the object were to be in contact with the plate throughout the motion?

*Solution.* (i) The object slides when the static friction is no longer strong enough to ensure the acceleration of the object matches the acceleration of the platform.

The maximum static friction between the object and platform is  $f = \mu mg$ . For SHM, the maximum acceleration is  $a_{max} = \omega^2 A$ . Hence,

$$\mu mg = m\omega^2 A \quad \Rightarrow \quad \mu = \frac{\omega^2 A}{g} = \frac{\left(\frac{2\pi}{T}\right)^2 A}{g} = 0.40$$

(ii) To stay in contact, the normal force must be positive. At the extreme case, we may set the normal force to be 0, thus the maximum acceleration of the block must be equal to  $-g$  at the highest point. Hence,

$$\omega^2 A = g \quad \Rightarrow \quad A = \frac{g}{\omega^2} = g \left(\frac{T}{2\pi}\right)^2 = 0.56 \text{ m}$$

**Problem 2.2** (SPhO 2011). A large block, with a second block sitting on top, is connected to a spring and executes horizontal simple harmonic motion as it slides across a frictionless surface with an angular frequency  $\omega$ . The coefficient of static friction between the two blocks is  $\mu_s$ . Determine the maximum amplitude of oscillation if the upper block is not to slip.

*Solution.* Since the blocks are moving together, it is essentially a spring-mass system with mass  $M + m$ , where  $M$  and  $m$  are the masses of the large and small blocks respectively.

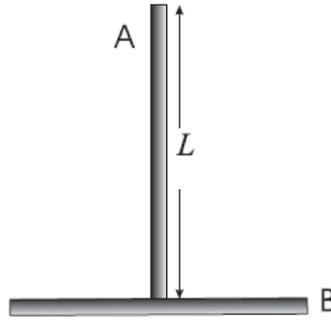
Thus, the angular frequency of oscillation is

$$\omega = \sqrt{\frac{k}{M_{tot}}} = \sqrt{\frac{k}{M + m}}$$

The maximum acceleration of the top block is  $a_{max} = \omega^2 A$ . This corresponds to the maximum frictional force  $f_{max} = \mu_s Mg$ . Hence, the maximum amplitude is

$$A = \frac{\mu_s g}{\omega^2}$$

**Problem 2.3** (SPhO 2004). A T-shaped structure is formed from two uniform rods,  $A$  and  $B$ , each of length  $L$  and mass  $m$ , as shown below. A point mass  $m$  moves horizontally at right angles to rod  $A$ , strikes the end of rod  $B$  with an initial velocity  $V$  and sticks to it. (i) Find the position of the point on the T-shaped structure which remains stationary. (ii) Determine the angular velocity of the system immediately after the collision. (iii) Sketch the positions of the T-shaped structure in its subsequent motion. (iv) Determine the change in kinetic energy of the system as a result of the collision. (v) The point mass  $m$  is now removed, and the T-shaped structure is suspended from the free end of rod  $A$  and allowed to move freely in the plane of the "T". Determine the period of small oscillations.



*Solution.* Let us define a coordinate system where the point of intersection of the rods is at the origin  $(0, 0)$ . We can calculate the position of the CM right after collision:

$$x_{CM} = \frac{m\left(-\frac{L}{2}\right)}{3m} = -\frac{L}{6}, \quad y_{CM} = \frac{m\left(\frac{L}{2}\right)}{3m} = \frac{L}{6}$$

and, applying the parallel axis theorem, the MOI of the system about the CM is

$$\begin{aligned} I_{CM} &= \frac{1}{12}mL^2 + m\left(\left(-\frac{L}{6}\right)^2 + \left(\frac{L}{6}\right)^2\right) + \frac{1}{12}mL^2 + m\left(\left(\frac{L}{6}\right)^2 + \left(\frac{L}{3}\right)^2\right) + m\left(\left(\frac{L}{6}\right)^2 + \left(\frac{L}{3}\right)^2\right) \\ &= \frac{1}{2}mL^2 \end{aligned}$$

(i) As no external force acts on the rods+mass system, by COM,

$$mV = 3mv_{CM} \quad \Rightarrow \quad v_{CM} = \frac{V}{3}$$

As no external torque acts on the rods+mass system, by COAM about the CM,

$$mV\left(\frac{L}{6}\right) = I_{CM}\omega_{CM} = \frac{1}{2}mL^2\omega_{CM} \quad \Rightarrow \quad \omega_{CM} = \frac{V}{3L} = \frac{v_{CM}}{L}$$

The point that remains stationary is the ICOR. At this point, the velocity vectors due to  $v_{CM}$  and  $r\omega_{CM}$  must cancel out, where  $r$  is the distance from the CM to that point.

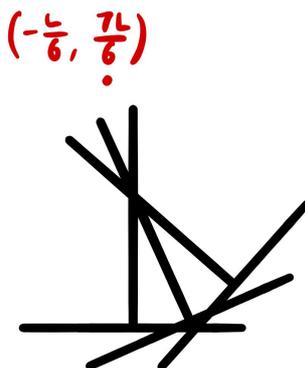
Clearly, this point is  $L$  above the ICOR in the  $y$ -direction. Thus,

$$\text{Position of ICOR: } \left(-\frac{L}{6}, \frac{7L}{6}\right)$$

(ii) From above, we already found that

$$\omega_{CM} = \frac{V}{3L}$$

(iii) A sketch that shows that the whole structure is rotating about the ICOR is correct.



(iv) The change in KE is just

$$\Delta K = \frac{1}{2}(3m)v_{CM}^2 + \frac{1}{2}I_{CM}\omega_{CM}^2 - \frac{1}{2}mV^2 = -\frac{11}{36}mV^2$$

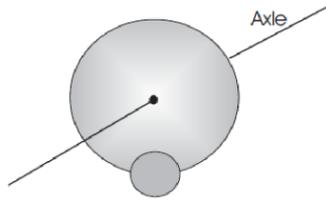
(v) The MOI of the structure about the pivot (which is the free end of rod  $A$ ) is

$$I_A = \frac{1}{3}mL^2 + \frac{1}{12}mL^2 + mL^2 = \frac{17}{12}mL^2$$

The CM is at a distance  $d = \frac{3L}{4}$  from this point. Thus, using the period of a physical pendulum,

$$T = 2\pi\sqrt{\frac{(2m)g\left(\frac{3L}{4}\right)}{\frac{17}{12}mL^2}} = 2\pi\sqrt{\frac{17L}{18g}}$$

**Problem 2.4** (SPhO 2005). A small thin disk of radius  $r$  and mass  $m$  is **attached rigidly** to the face of a second thin disk of radius  $R$  and mass  $M$ , as shown below. The centre of the small disk is located at the edge of the large disk. The large disk is mounted at its centre on a frictionless horizontal axle. The assembly is rotated through an angle  $\theta$  from its equilibrium position and released. (i) Find the speed of the centre of the small disk as it passes through the equilibrium position. (ii) Determine the period of small oscillations.



*Solution.* You actually need some grasp of rotation to solve this problem.

(i) The MOI of the whole set-up about the axle is

$$I = I_R + I_r = \frac{1}{2}MR^2 + \left(\frac{1}{2}mr^2 + mR^2\right)$$

where we invoked the parallel axis theorem for the small disk.

Equating the change in GPE to the change in KE (accounting for rotational KE),

$$mgR(1 - \cos\theta) = \frac{1}{2}I\omega^2 \quad \Rightarrow \quad \omega = \sqrt{\frac{2mgR(1 - \cos\theta)}{I}} = \sqrt{\frac{2mgR(1 - \cos\theta)}{\frac{1}{2}MR^2 + \frac{1}{2}mr^2 + mR^2}}$$

(ii) For small oscillations,  $\theta$  is small. Thus, using  $\cos\theta \approx 1 - \frac{\theta^2}{2}$ , the GPE at angle  $\theta$  is

$$U_g = mgR(1 - \cos\theta) \approx \frac{1}{2}mgR\theta^2 = \frac{1}{2}k_{eff}\theta^2, \quad k_{eff} = mgR$$

The rotational KE remains as

$$K = \frac{1}{2}I\omega^2 = \frac{1}{2}I\dot{\theta}^2 = \frac{1}{2}m_{eff}\dot{\theta}^2, \quad m_{eff} = I$$

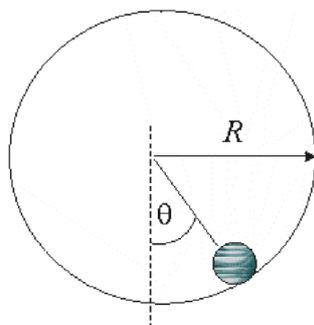
Hence, the angular frequency of small oscillations is

$$\Omega = \sqrt{\frac{k_{eff}}{m_{eff}}} = \sqrt{\frac{mgR}{I}} = \sqrt{\frac{mgR}{\frac{1}{2}MR^2 + \frac{1}{2}mr^2 + mR^2}}$$

and the period is hence

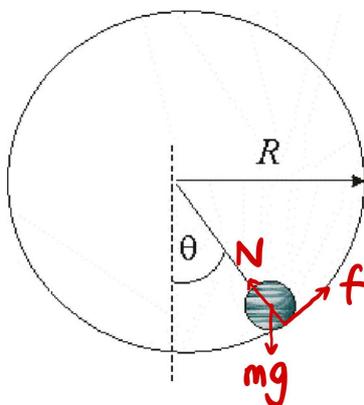
$$T = \frac{2\pi}{\Omega} = 2\pi \sqrt{\frac{\frac{1}{2}MR^2 + \frac{1}{2}mr^2 + mR^2}{mgR}}$$

**Problem 2.5** (SPhO 2002). A solid sphere with radius  $r$  and mass  $m$  rotates inside a **fixed** hollow sphere of radius  $R$ , as shown below. Calculate the period of small oscillations about the equilibrium, assuming that the sphere rolls without slipping. The moment of inertia of the solid sphere about its central axis is  $\frac{2}{5}mr^2$ . (You'll need to know some rotation for this problem.)



*Solution.* Since we presented the energy method for the last problem, let's present the force method for this one.

The forces acting on the sphere are labelled as such:



Writing N2L in the tangential direction and using small angle approximation,

$$mg \sin \theta - f = ma \quad \Rightarrow \quad mg\theta - f = ma$$

Writing the net torque on the sphere,

$$fr = I\alpha = \frac{2}{5}mr^2\alpha \quad \Rightarrow \quad f = \frac{2}{5}mr\alpha$$

The non-slip condition is

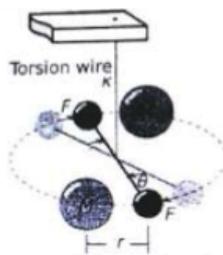
$$a = (R - r)\alpha$$

Combining all these results, and being careful with signs, you will get

$$\ddot{\theta} + \frac{5g}{7(R-r)}\theta = 0 \quad \Rightarrow \quad \omega = \sqrt{\frac{5g}{7(R-r)}} \quad \Rightarrow \quad T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{7(R-r)}{5g}}$$

**Problem 2.6.** In the Cavendish experiment the two *small* balls have mass  $m$  each and are connected by a light rigid rod of length  $L$ . Two *large* balls have mass  $M$  each and their centres are also separated by the same distance  $L$ . The torsion constant of the suspension fibre is  $\kappa$ .

- (a) *Large balls absent.* Find the period of the torsion pendulum.
- (b) The large balls are placed a small distance away from the small balls. After a static equilibrium is reached the rod has rotated through a small angle  $\theta$  and the centre-to-centre distance of each pair is  $r$  (see figure). Derive an expression for the gravitational constant  $G$  in terms of  $L$ ,  $r$ ,  $\theta$  and the period  $T$  from part (a).
- (c) If the small balls are perturbed through a *small* angle about the equilibrium in (b), determine whether the motion is simple harmonic when first-order changes in the gravitational force are included, and obtain the resulting period.



*Solution.* We first find the moment of inertia of the system. Each small ball is at distance  $L/2$  from the torsion fibre, hence

$$I = 2m\left(\frac{L}{2}\right)^2 = \frac{mL^2}{2}.$$

- (a) For torsional motion  $I\ddot{\theta} + \kappa\theta = 0$  so

$$T_0 = 2\pi\sqrt{\frac{I}{\kappa}} = 2\pi\sqrt{\frac{mL^2}{2\kappa}}.$$

- (b) The gravitational force between one large–small pair is  $F = GmM/r^2$ . Because it acts essentially perpendicular to the rod, each pair supplies a torque  $F(L/2)$ ; both pairs add, giving

$$\tau_g = \frac{GmML}{r^2}.$$

At equilibrium this equals the torsion torque,  $\kappa\theta = GmML/r^2$ . Eliminate  $\kappa$  via the period found in (a):  $\kappa = (2\pi/T_0)^2 I = (2\pi)^2 mL^2/(2T_0^2)$ . Substituting and solving for  $G$ ,

$$G = \frac{2\pi^2 L r^2}{M T_0^2} \theta.$$

- (c) This part is on small oscillations about a shifted equilibrium. Write the angular displacement as  $\theta(t) = \theta_0 + \varphi(t)$  with  $|\varphi| \ll \theta_0$ . Note here that  $\theta$  is defined with respect to the position of zero rotation of the suspension fibre, and  $\theta_0$  is the value of  $\theta$  calculated in (b), the equilibrium amount of rotation. A small additional rotation by  $\varphi$  shortens each separation by  $\Delta r \simeq -L\varphi$ , so

$$r(\theta) \simeq r - L\varphi.$$

The gravitational torque becomes

$$\tau_g(\theta) = \frac{GmML}{[r - L\varphi]^2} \approx \frac{GmML}{r^2} \left(1 + 2\frac{L\varphi}{r}\right),$$

where the binomial expansion  $(1 - x)^{-2} \simeq 1 + 2x$  ( $x \ll 1$ ) is used.

The net restoring torque is therefore

$$\tau_{\text{net}} = -\kappa(\theta_0 + \varphi) + \tau_g(\theta_0) + \frac{2GmML^2}{r^3}\varphi = -\left(\kappa - \frac{GmML^2}{r^3}\right)\varphi.$$

Using the equilibrium condition  $\kappa\theta_0 = GmML/r^2$  to cancel constants,

$$\kappa - \frac{GmML^2}{r^3} = \kappa\left(1 - \frac{L\theta_0}{r}\right).$$

Hence

$$I\ddot{\varphi} + \kappa\left(1 - \frac{L\theta_0}{r}\right)\varphi = 0 \quad \implies \quad \omega^2 = \frac{\kappa}{I}\left(1 - \frac{L\theta_0}{r}\right).$$

The period becomes

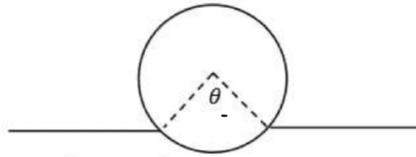
$$T = 2\pi\sqrt{\frac{I}{\kappa\left(1 - \frac{L\theta_0}{r}\right)}} = 2\pi\sqrt{\frac{mL^2r}{2\kappa(r - L\theta_0)}}$$

which, replacing the theoretical  $\theta_0$  by the measured deflection  $\theta$ , gives:

$$T = 2\pi\sqrt{\frac{mL^2r}{2\kappa(r - L\theta)}}$$

Thus the motion about the shifted equilibrium *is* simple harmonic, but with a slightly lengthened period due to the finite gravitational interaction.

**Problem 2.7.** A solid cylinder of radius  $a$  and length  $\ell$ , made of a lightweight material of uniform density  $d$ , is floating in seawater of density  $\rho$ . The gravitational acceleration is  $g$ . If the system is in equilibrium, write the expression relating the equilibrium tilt angle  $\theta$  to the parameters  $a$ ,  $\ell$ ,  $d$ ,  $\rho$ , and  $g$ . The cylinder is then given a small downward perturbation from equilibrium, find the frequency of the resulting small oscillations. You may express your answer in terms of the equilibrium angle  $\theta$ .



*Solution.* This problem is modified from IPhO 1995.

(a) Equilibrium immersion angle

We first find the submerged volume. For a circular segment of central angle  $\theta$ , the cross-sectional area below the water line is

$$A(\theta) = \frac{a^2}{2} (\theta - \sin \theta).$$

Hence the displaced volume is

$$V(\theta) = A(\theta) \ell = \frac{a^2 \ell}{2} (\theta - \sin \theta).$$

The cylinder's weight and the buoyant force must balance:

$$\underbrace{d\pi a^2 \ell g}_{\text{weight}} = \underbrace{\rho g V(\theta)}_{\text{buoyant force}} = \rho g \frac{a^2 \ell}{2} (\theta - \sin \theta).$$

Cancelling the common factors  $a^2 \ell g$  gives the relation

$$\boxed{2d\pi = \rho(\theta - \sin \theta)}.$$

(b) Frequency of small vertical oscillations

We now give the cylinder a small vertical displacement  $y$  (positive downwards) from its equilibrium position, keeping it horizontal at all times. Both the immersed volume and the buoyant force change, giving a restoring force.

For central angle  $\theta$  the water line (a horizontal chord) lies a vertical distance

$$h = a \cos\left(\frac{\theta}{2}\right)$$

above the cylinder's centre. A small downward shift  $y$  *reduces* this distance, so

$$dh = -dy, \quad \frac{dh}{d\theta} = -\frac{a}{2} \sin\left(\frac{\theta}{2}\right) \implies d\theta = \frac{2 dy}{a \sin(\theta/2)}.$$

Differentiate the displaced area:

$$\frac{dA}{d\theta} = \frac{a^2}{2} (1 - \cos \theta) = a^2 \sin^2\left(\frac{\theta}{2}\right).$$

Hence

$$\frac{dA}{dy} = \frac{dA}{d\theta} \frac{d\theta}{dy} = a^2 \sin^2\left(\frac{\theta}{2}\right) \cdot \frac{2}{a \sin(\theta/2)} = 2a \sin\left(\frac{\theta}{2}\right).$$

Because  $V = A\ell$ , the rate at which displaced volume changes with  $y$  is

$$\frac{dV}{dy} = 2a\ell \sin\left(\frac{\theta}{2}\right),$$

so the incremental buoyant force is

$$\frac{dF_b}{dy} = \rho g \frac{dV}{dy} = 2\rho g a \ell \sin\left(\frac{\theta}{2}\right).$$

Now we want to get the equation of motion. For small oscillations ( $y \ll a$ ) the net restoring force is linear:

$$F_{\text{net}} \approx -\frac{dF_b}{dy} y = -k y, \quad k = 2\rho g a \ell \sin\left(\frac{\theta}{2}\right).$$

With mass  $m = d\pi a^2 \ell$ , N2L gives

$$m\ddot{y} + ky = 0 \implies \omega^2 = \frac{k}{m} = \frac{2\rho g a \ell \sin(\theta/2)}{d\pi a^2 \ell} = \frac{2\rho g}{d\pi a} \sin\left(\frac{\theta}{2}\right).$$

$$\boxed{f = \frac{\omega}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{2\rho g}{d\pi a} \sin\left(\frac{\theta}{2}\right)}}.$$

**Problem 2.8** (SPhO 2014). Problems involving springs often consider the springs to be massless. Of course, this is not true in reality. Here, we consider a spring of mass  $M$ , equilibrium length  $L_0$  and spring constant  $k$ . One end of the spring is fixed to a wall, and a mass  $m$  is attached at the other end of the spring. Neglecting gravity, determine the period of oscillations of this system. (Hint: Consider energy. How do you find the kinetic energy of the spring?)

*Solution.* As mentioned in the hint, we shall calculate the total KE of the system, to find  $m_{\text{eff}}$ .

One end of the spring is fixed to a wall and stationary, while the other end of the spring is moving at speed  $v$ . For a linear spring of current length  $L$ , the speed of a point at a distance  $x$  (where  $0 \leq x \leq L$ ) is given by

$$v_x = \frac{vx}{L}$$

Thus, the infinitesimal KE of one segment is

$$dK = \frac{1}{2} (dm) v_x^2 = \frac{1}{2} \left(\frac{M}{L} dx\right) \left(\frac{vx}{L}\right)^2$$

Integrating across the whole spring,

$$K_{\text{spring}} = \int dK = \int_0^L \frac{1}{2} \left(\frac{M}{L} dx\right) \left(\frac{vx}{L}\right)^2 = \frac{Mv^2}{2L^3} \int_0^L x^2 dx = \frac{Mv^2}{2L^3} \left(\frac{L^3}{3}\right) = \frac{1}{6} Mv^2$$

Thus, the total KE of the system is

$$K = \frac{1}{6} Mv^2 + \frac{1}{2} mv^2 = \frac{1}{2} \left(m + \frac{M}{3}\right) v^2 = \frac{1}{2} m_{\text{eff}} v^2, \quad m_{\text{eff}} = m + \frac{M}{3}$$

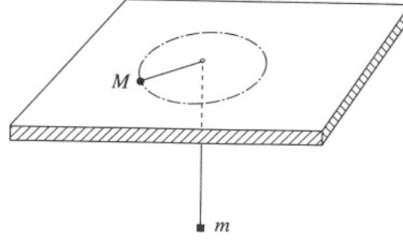
Thus, the period of oscillations can be determined:

$$\omega = \sqrt{\frac{k}{m_{\text{eff}}}} = \sqrt{\frac{k}{m + \frac{M}{3}}} \implies T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{m + \frac{M}{3}}{k}}$$

**Remark.** This result only holds well when  $M \ll m$ , but this is physically sound anyway (you'd expect springs to be very light).

**Problem 2.9** (Cahn). A particle of mass  $M$  is constrained to move on a frictionless horizontal plane. A second particle of mass  $m$  is constrained to a vertical line. The two particles are connected by a massless string which passes through a hole in the plane. The system is set-up so that the mass  $M$  moves in a circle of radius  $r$ , while the mass  $m$  remains still. Show that

this motion is stable with respect to small changes in  $r$ , and find the angular frequency of small oscillations.



*Solution.* Firstly, the angular momentum of mass  $M$  is conserved, since the tension acts radially and hence there is no net external torque. Thus,

$$L = Mr^2\dot{\theta} \quad \Rightarrow \quad \dot{\theta} = \frac{L}{Mr^2}$$

At equilibrium,

$$T = Mr\omega^2 = mg \quad \Rightarrow \quad \omega = \sqrt{\frac{mg}{Mr}} \quad \Rightarrow \quad L = mr^2\omega = mr^2\sqrt{\frac{mg}{Mr}}$$

Writing N2L in the radial direction for  $M$ ,

$$-T = M(\ddot{r} - r\dot{\theta}^2) = M\left(\frac{L^2}{M^2r^3} - \ddot{r}\right)$$

Be careful with the signs here! The net force is  $-T$  as tension acts radially *inward*.

Writing N2L in the vertical direction for  $m$ ,

$$T - mg = m\ddot{r} \quad \Rightarrow \quad T = mg + m\ddot{r}$$

Again, be careful with the signs here!  $mg$  decreases  $r$  while  $T$  increases  $r$ , which explains the signs in the N2L equation.

Thus, substituting  $T$  into the other equation,

$$mg + m\ddot{r} = \frac{L^2}{Mr^3} - Mr\ddot{r} \quad \Rightarrow \quad (M + m)\ddot{r} = \frac{L^2}{Mr^3} - mg$$

Now, we imagine replacing  $r$  with  $r + \delta$ , where  $\delta \ll r$ . Then,  $\ddot{\delta} = \ddot{r}$ . Thus,

$$(M + m)\ddot{\delta} = \frac{L^2}{M}(r + \delta)^{-3} - mg = \frac{L^2}{Mr^3} \left(1 + \frac{\delta}{r}\right)^{-3} - mg \approx \frac{L^2}{Mr^3} \left(1 - \frac{3\delta}{r}\right) - mg$$

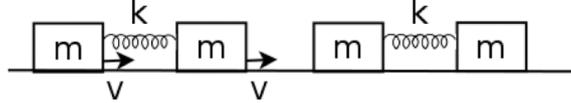
Rearranging, we find that

$$(M + m)\ddot{\delta} + \frac{3L^2}{Mr^4}\delta = \frac{L^2}{Mr^3} - mg$$

Hence, this is a SHM with angular frequency

$$\Omega = \sqrt{\frac{3L^2}{Mr^4(M + m)}} = \sqrt{\frac{3}{Mr^4(M + m)} \left(M^2r^4 \frac{mg}{Mr}\right)} = \sqrt{\frac{3mg}{(M + m)r}}$$

**Problem 2.10.** Two point masses of mass  $m$  are connected by a spring of spring constant  $k$  and relaxed length  $\ell$ . The two masses both have an initial velocity  $v$  and the spring between them remains at its relaxed length. These masses then travel towards an identical setup (consisting of two masses connected by a spring) on a frictionless, horizontal table. If these four masses are aligned and undergo perfectly elastic, head-on collisions:



1. Determine the equations of motion of the masses after the first collision and before the second collision.
2. Determine the elapsed time between the first and second collisions, and show that there will be only a total of two collisions.

*Solution.* Because all four masses are identical and the first impact between the inner two masses is perfectly elastic, they simply exchange velocities. Let the masses be numbered left to right  $m_1, m_2, m_3, m_4$  and let the positive  $\hat{x}$ -direction point to the right. Just after the first collision we have

$$\dot{x}_1 = v, \quad \dot{x}_2 = 0, \quad \dot{x}_3 = v, \quad \dot{x}_4 = 0.$$

The two systems thus look identical, with both left masses possessing velocity  $v$  and right masses with zero velocity. Consider the left system ( $m_1, m_2$ ); the right one behaves identically. Introduce

$$\begin{aligned} y_1 &\equiv x_1 \quad (\text{left mass}), & y_2 &\equiv x_2 \quad (\text{right mass}), \\ u &\equiv y_2 - y_1 - \ell \quad (\text{spring extension from rest}). \end{aligned}$$

Hooke's law supplies equal and opposite internal forces  $\pm ku$  so that

$$m\ddot{y}_1 = +ku, \quad m\ddot{y}_2 = -ku.$$

Subtracting gives the *relative* equation of motion

$$\boxed{\ddot{u} + \omega^2 u = 0}, \quad \omega \equiv \sqrt{\frac{2k}{m}}$$

Right after  $t = 0$

$$u(0) = 0, \quad \dot{u}(0) = \dot{y}_2 - \dot{y}_1 = 0 - v = -v.$$

Solving the SHM with these data yields

$$u(t) = -\frac{v}{\omega} \sin(\omega t) = -v \sqrt{\frac{m}{2k}} \sin\left(\sqrt{\frac{2k}{m}} t\right).$$

The next collision occurs when the inner two masses again meet, i.e.  $y_2 = y_3$ . For the left dimer that translates to  $u = -\ell$  (a full spring compression). Setting  $u(t_c) = -\ell$  gives

$$-\ell = -\frac{v}{\omega} \sin(\omega t_c) \implies \sin(\omega t_c) = \frac{\ell \omega}{v}.$$

But the two dimers were initially spaced by exactly their relaxed length  $\ell$ , so  $\ell \omega / v = 1$ ; therefore

$$\boxed{t_c = \frac{\pi}{\omega} = \pi \sqrt{\frac{m}{2k}}}.$$

Insert  $t_c$  into the centre-of-mass and relative motions:

$$\dot{y}_1(t) = v + \frac{v}{2} \cos(\omega t), \quad \dot{y}_2(t) = 0 - \frac{v}{2} \cos(\omega t).$$

At  $t = t_c$ ,  $\cos(\omega t_c) = -1$ , giving

$$\dot{y}_1 = 0, \quad \dot{y}_2 = v \quad (\text{just before impact}),$$

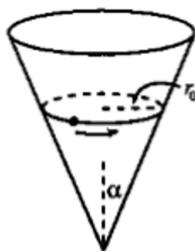
with the right dimer yielding the mirror-image ( $\dot{x}_3 = 0, \dot{x}_4 = v$ ).

A second elastic collision *again* swaps the velocities of the inner pair. Hence

$$\dot{x}_1 = 0, \quad \dot{x}_2 = v, \quad \dot{x}_3 = 0, \quad \dot{x}_4 = v \quad (\text{after impact}).$$

The initial kinetic energy of masses  $m_1$  and  $m_2$  has been fully handed off to  $m_2$  and  $m_4$ . The two moving masses now recede from one another, so no further collisions are possible.

**Problem 2.11** (Ricardo). A particle moves on a frictionless cone. The cone is fixed with its tip on the ground and its axis vertical. The cone has a half-angle of  $\alpha$ . Let the particle move in a circle of radius  $r_0$ . (i) Find the frequency  $\omega$  of the circular motion. (ii) Let the particle be perturbed slightly from this motion. Find the frequency  $\Omega$  of oscillations about the radius  $r_0$ .



*Solution.* You need prior knowledge of rotation and angular momentum to solve this problem.

(i) Normal and gravitational forces act on the particle. Writing N2L, we have

$$N \sin \alpha = mg, \quad N \cos \alpha = mr_0 \omega^2 \quad \Rightarrow \quad \tan \alpha = \frac{g}{r_0 \omega^2} \quad \Rightarrow \quad \omega = \sqrt{\frac{g}{r_0 \tan \alpha}}$$

(ii) Here, it is much easier to use energy. We write the KE and GPE:

$$K = \frac{1}{2} m v^2 = \frac{1}{2} m \left( (r\dot{\theta})^2 + \left( \frac{\dot{r}}{\sin \alpha} \right)^2 \right), \quad U = \frac{mgr}{\tan \alpha}$$

where we have worked in polar coordinates,  $r$  and  $\theta$ .

Additionally, angular momentum of the particle is conserved, since no net external torque acts on it:

$$L = mr^2 \dot{\theta} \quad \Rightarrow \quad \dot{\theta} = \frac{L}{mr^2}$$

Substituting this into the KE expression,

$$K = \frac{1}{2} m \left( \frac{L^2}{m^2 r^2} + \frac{\dot{r}^2}{\sin \alpha} \right)$$

Now, since total energy is conserved as there are no dissipative forces,

$$\frac{dE}{dt} = 0 \Rightarrow \frac{d}{dt}(K + U) = 0 \Rightarrow \frac{d}{dt} \left( \frac{1}{2} m \left( \frac{L^2}{m^2 r^2} + \frac{\dot{r}^2}{\sin^2 \alpha} \right) + \frac{mgr}{\tan \alpha} \right) = 0$$

$$\ddot{r} = \left( \frac{L^2}{m^2 r^3} - \frac{g}{\tan \alpha} \right) \sin^2 \alpha$$

Now, we shall use the small perturbation method, as introduced earlier. Set  $r = r_0 + \delta$ , where  $\delta \ll r_0$ . Then,  $\ddot{r} = \ddot{\delta}$ . We can also perform binomial expansion:

$$\frac{L^2}{m^2 (r_0 + \delta)^3} = \frac{L^2}{m^2} (r_0 + \delta)^{-3} = \frac{L^2}{m^2 r_0^3} \left( 1 + \frac{\delta}{r_0} \right)^{-3} \approx \frac{L^2}{m^2 r_0^3} \left( 1 - \frac{3\delta}{r_0} \right)$$

Let's start to simplify everything. The angular momentum is

$$L = mr_0 v = mr_0^2 \omega = mr_0^2 \sqrt{\frac{g}{r_0 \tan \alpha}} \Rightarrow \frac{L^2}{m^2 r_0^3} = \frac{g}{\tan \alpha}$$

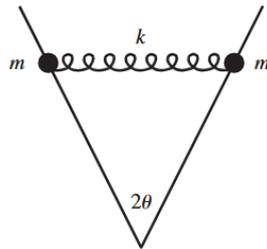
Thus, substituting this into the earlier expression,

$$\ddot{\delta} = -\frac{3g \sin^2 \alpha}{r_0 \tan \alpha} \delta = -\left( \frac{3g}{r_0} \sin \alpha \cos \alpha \right) \delta \Rightarrow \ddot{\delta} + \left( \frac{3g}{r_0} \sin \alpha \cos \alpha \right) \delta = 0$$

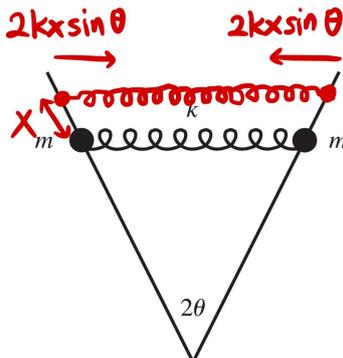
Hence, this is a SHM with angular frequency

$$\Omega = \sqrt{\frac{3g}{r_0} \sin \alpha \cos \alpha}$$

**Problem 2.12** (Morin). Two particles of mass  $m$  are constrained to move along two horizontal frictionless rails that make an angle  $2\theta$  with respect to each other. They are connected by a spring with spring constant  $k$ , whose relaxed length is shown below. What is the frequency of oscillations for the motion where the spring remains parallel to the position shown?



*Solution.* Let  $x$  be the distance that each mass has moved along the rail, relative to equilibrium.



The spring force  $2kx \sin \theta$  acts on each mass, in the direction parallel to the spring. Thus, the component of the force *along* the rail is  $F_{rail} = 2kx \sin^2 \theta$ .

Writing N2L, we have

$$m\ddot{x} = -2kx \sin^2 \theta \quad \Rightarrow \quad \ddot{x} + \frac{2k \sin^2 \theta}{m} x = 0 \quad \Rightarrow \quad \omega = \sqrt{\frac{2k \sin^2 \theta}{m}} = \sqrt{\frac{2k}{m}} \sin \theta$$

### 3 Advanced Problems

**Problem 3.1.** A spring with relaxed length zero and spring constant  $k$  is attached to the ground. A projectile of mass  $m$  is attached to the other end of the spring. The projectile is then picked up and thrown with velocity  $v$  at an angle  $\theta$  to the horizontal.

1. Geometrically, what kind of curve is the resulting trajectory?
2. Find the value of  $v$  so that the projectile hits the ground traveling straight downward.

*Solution.* Let the anchor of the spring be at the origin. The total force on the particle is

$$\mathbf{F} = -k\mathbf{r} - mg\hat{\mathbf{y}} = -k(\mathbf{r} - \mathbf{r}_0), \quad \text{where } \mathbf{r}_0 = \frac{mg}{k}\hat{\mathbf{y}}.$$

Thus the particle experiences an effective spring force about equilibrium at  $\mathbf{r}_0$ . In two dimensions each of  $x$  and  $y$  executes simple harmonic motion of the same frequency  $\omega = \sqrt{k/m}$ , so the trajectory is a portion of an ellipse whose center is displaced by  $mg/k$  below the launch point. In the horizontal direction,

$$v_x(t) = (v \cos \theta) \cos(\omega t),$$

which first vanishes when  $\omega t = \pi/2$ , i.e. after a quarter period. At that instant the particle returns to ground level, so its vertical displacement from equilibrium must be zero. Vertically the motion is simple harmonic about  $y_{\text{eq}} = mg/k$ , so write

$$v_y(t) = v_0 \cos(\omega t + \phi).$$

The initial phase is  $\phi$ , and just before impact the vertical velocity is reversed in sign, corresponding to phase  $\pi - \phi$ . Since impact occurs a quarter period later,

$$(\pi - \phi) - \phi = \frac{\pi}{2} \implies \phi = \frac{\pi}{4}.$$

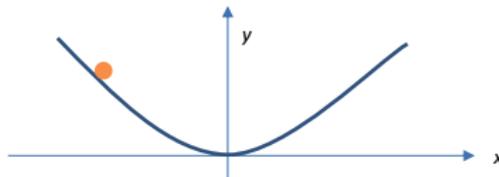
Matching  $v_y(0) = v \sin \theta$  and  $\dot{v}_y(0) = -g$  gives

$$v_0 \cos \phi = v \sin \theta, \quad -v_0 \omega \sin \phi = -g.$$

Dividing these equations,

$$\tan \phi = \frac{g}{\omega v \sin \theta}, \quad \text{and since } \tan \phi = 1, \quad v = \frac{g}{\omega \sin \theta} = \frac{g}{\sqrt{k/m} \sin \theta} = \boxed{\sqrt{\frac{m}{k}} \frac{g}{\sin \theta}}.$$

**Problem 3.2** (SPOT TST 2017). A point object of mass  $m$  slides on a frictionless parabolic curve in a uniform gravitational field  $g$ . The curve shown below has equation  $y = \frac{x^2}{L}$  where  $L$  is a constant.



1. (a) Derive an expression for the total energy  $E_{\text{parabolic}}$  of the point object, as a function only of  $x$  and  $\dot{x}$ , where  $\dot{x} = \frac{dx}{dt}$ .  
 (b) Show that the motion of the point object is only *approximately* simple harmonic, and find the period  $T_{\text{approx}}$ .

2. Instead of the frictionless parabolic curve, we wish to find the frictionless tautochrone<sup>1</sup> curve, on which a point object will oscillate with the same period regardless of where it is released from. Let this common period be  $T_0$ .
- (a) Define the arc length  $s$  as the distance along the curve from the origin. Derive an expression for the total energy  $E_{\text{tautochrone}}$  of the point object, written as a function only of  $s$  and  $\dot{s}$ .
- (b) Deduce the parametric equations defining the tautochrone curve, i.e. expressions  $x(\theta)$ ,  $y(\theta)$  where the angle  $\theta$  is related to the slope of the curve by

$$\tan \theta = \frac{dy}{dx}.$$

*Solution.* We deal with the cases separately

### 1. Motion on the parabolic track $y = x^2/L$

- (a) *Total energy in terms of  $x$  and  $\dot{x}$ .*

Along the curve  $y(x) = x^2/L$ ,

$$\dot{y} = \frac{dy}{dx} \dot{x} = \frac{2x}{L} \dot{x}, \quad v^2 = \dot{x}^2 + \dot{y}^2 = \dot{x}^2 \left(1 + \frac{4x^2}{L^2}\right).$$

Hence

$$E_{\text{parabolic}}(x, \dot{x}) = \frac{1}{2} m \dot{x}^2 \left(1 + \frac{4x^2}{L^2}\right) + \frac{mg}{L} x^2 \quad (y(0) = 0).$$

- (b) *Small-amplitude approximation.*

For  $|x| \ll L$  the factor  $(1 + 4x^2/L^2)$  is  $\approx 1$  and the *Lagrangian*  $L = T - V$  may be truncated at first order in  $x$ :

$$L \approx \frac{1}{2} m \dot{x}^2 - \frac{mg}{L} x^2.$$

The Euler–Lagrange equation then reads

$$m\ddot{x} + \frac{2mg}{L} x = 0 \implies \ddot{x} + \omega^2 x = 0, \quad \omega = \sqrt{\frac{2g}{L}}.$$

Thus the bead executes *approximate* simple harmonic motion with period

$$T_{\text{approx}} = 2\pi \sqrt{\frac{L}{2g}}.$$

### 2. Tautochrone (cycloidal) track with common period $T_0$

Let  $s$  denote arc length measured from the lowest point ( $s = 0$ ).

- (a) *Energy in terms of  $s$  and  $\dot{s}$ .*

To obtain a *constant* period  $T_0$  for all amplitudes, the motion in  $s$  must be strictly simple harmonic:

$$\ddot{s} + \omega^2 s = 0, \quad \omega = \frac{2\pi}{T_0}.$$

<sup>1</sup>On a tautochrone curve, a point object will oscillate with the same period regardless of its release point (i.e. period independent of amplitude, just like a simple harmonic oscillator).

Projecting  $mg$  along the tangent ( $\sin \theta$  component) gives  $m\ddot{s} = -mg \sin \theta \implies g \sin \theta = \omega^2 s$ . Since  $dy/ds = \sin \theta$ , integrate to find  $y(s) = (\omega^2/2g) s^2$ . Therefore

$$E_{\text{tauto}}(s, \dot{s}) = \frac{1}{2}m\dot{s}^2 + \frac{1}{2}m\omega^2 s^2.$$

(b) *Parametric form of the curve.*

From  $g \sin \theta = \omega^2 s$  we have  $s = (g/\omega^2) \sin \theta$ ,  $ds/d\theta = (g/\omega^2) \cos \theta$ . Using  $ds^2 = dx^2 + dy^2$  and  $\tan \theta = dy/dx$ :

$$\frac{dx}{d\theta} = \cos \theta \frac{ds}{d\theta} = \frac{g}{\omega^2} \cos^2 \theta, \quad \frac{dy}{d\theta} = \tan \theta \frac{dx}{d\theta} = \frac{g}{\omega^2} \sin \theta \cos \theta.$$

Integrating from  $\theta = 0$  (lowest point) and defining  $R = g/(4\omega^2) = gT_0^2/(16\pi^2)$ ,

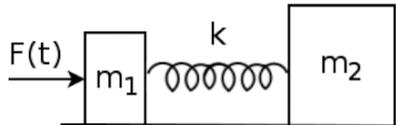
$$\begin{aligned} x(\theta) &= R(2\theta + \sin 2\theta), \\ y(\theta) &= R(1 - \cos 2\theta). \end{aligned}$$

These equations describe an *inverted cycloid*, the classical tautochrone first discovered by Huygens; a bead released from any height reaches the bottom in the same time  $T_0$ .

**Problem 3.3.** A heavy bob is often used to stabilize buildings in the event of earthquakes. Let us consider a related problem. Two masses  $m_1$  and  $m_2$  are stationary on a horizontal, frictionless plane and are connected by a spring of spring constant  $k$ . Suppose a force

$$F(t) = f \cos \omega t$$

is exerted on  $m_1$ , in the direction of the line joining the two masses. Determine the value of  $k$  for which the particular solution to the equation of motion of  $m_1$  yields an oscillation of *zero amplitude*. There is no damping.



*Solution.* Since the motion is purely one-dimensional, choose the  $x$ -axis along the line of motion and let  $x_1(t)$  and  $x_2(t)$  be the displacements of masses  $m_1$  and  $m_2$  from their respective equilibrium positions. The relaxed length of the spring can be taken as zero. The equations of motion are

$$m_1 \ddot{x}_1 = f \cos \omega t + k(x_2 - x_1), \quad m_2 \ddot{x}_2 = -k(x_2 - x_1).$$

Multiply the second equation by  $m_1$  and subtract the first multiplied by  $m_2$  to eliminate the spring term:

$$m_1 m_2 (\ddot{x}_2 - \ddot{x}_1) = -k(m_1 + m_2)(x_2 - x_1) - m_2 f \cos \omega t.$$

Define the relative coordinate  $u = x_2 - x_1$ . Then

$$\ddot{u} + \frac{k(m_1 + m_2)}{m_1 m_2} u = -\frac{f}{m_1} \cos \omega t.$$

To find the particular solution, replace  $\cos \omega t$  by the real part of  $e^{i\omega t}$  and solve

$$\ddot{u} + \Omega^2 u = -\frac{f}{m_1} e^{i\omega t}, \quad \Omega^2 = \frac{k(m_1 + m_2)}{m_1 m_2}.$$

Assume  $u = A e^{i\omega t}$ . Substitution gives

$$-A\omega^2 + \Omega^2 A = -\frac{f}{m_1} \implies A = \frac{m_2 f}{m_1 m_2 \omega^2 - k(m_1 + m_2)}.$$

Hence the real particular solution is

$$u(t) = A \cos \omega t.$$

Finally, substitute  $u$  back into the  $m_1 x_1$ -equation,

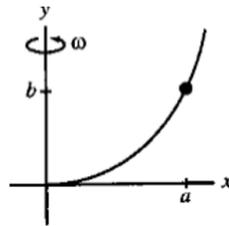
$$m_1 \ddot{x}_1 = f \cos \omega t + k u \implies \ddot{x}_1 = \frac{f}{m_1} \cos \omega t + \frac{k m_2 f}{m_1 (m_1 m_2 \omega^2 - k(m_1 + m_2))} \cos \omega t.$$

These two terms cancel when

$$\boxed{k = m_2 \omega^2}$$

yielding no net forced response of  $x_1$ .

**Problem 3.4** (Ricardo). The curve  $y = b \left(\frac{x}{a}\right)^\lambda$  is rotated around the  $y$ -axis with constant frequency  $\omega$ , and  $a, b > 0$ . A bead moves without friction along the curve. Find the frequency of small oscillations about the equilibrium point. Under what conditions do oscillations exist?



*Solution.* Let's present a solution that primarily uses energy. We go into the rotating frame of the curve.

In this frame, the bead only moves *along* the curve. Thus, the total KE is

$$K = \frac{1}{2} m v^2 = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) = \frac{1}{2} m \left( \dot{x}^2 + \dot{x}^2 \left( \frac{dy}{dx} \right)^2 \right)$$

whereby the last equality was obtained using the chain rule on  $\dot{y}$ .

We can calculate  $\frac{dy}{dx}$  using the equation of the curve:

$$\frac{dy}{dx} = \frac{b\lambda x^{\lambda-1}}{a^\lambda}$$

hence, the expression for  $K$  is

$$K = \frac{1}{2} m \left( \dot{x}^2 \left( 1 + \left( \frac{b\lambda x^{\lambda-1}}{a^\lambda} \right)^2 \right) \right)$$

For the total PE, recall that we must consider both gravitational and centrifugal PE. Thus,

$$U = mgy - \frac{1}{2} m \omega^2 x^2 = mgb \left( \frac{x}{a} \right)^\lambda - \frac{1}{2} m \omega^2 x^2$$

With this, we can answer the first part of the question. Equilibrium happens when  $\frac{dU}{dx} = 0$ , thus

$$\frac{dU}{dx} = \frac{mgb\lambda}{a^\lambda} x^{\lambda-1} - m\omega^2 x = 0 \quad \Rightarrow \quad x_0 = \left( \frac{\omega^2 a^\lambda}{\lambda gb} \right)^{\frac{1}{\lambda-2}}$$

To proceed with the second part, we can find  $k_{eff}$  using the second derivative of  $U$  at the equilibrium point:

$$\begin{aligned} \frac{d^2U}{dx^2} &= \frac{mgb\lambda(\lambda-1)}{a^\lambda} x^{\lambda-2} - m\omega^2 \\ k_{eff} &= \left. \frac{d^2U}{dx^2} \right|_{x=x_0} = m\omega^2(\lambda-1) - m\omega^2 = m\omega^2(\lambda-2) \end{aligned}$$

Going back to the kinetic energy, we can now find  $m_{eff}$  by setting  $K = \frac{1}{2}m_{eff}\dot{x}^2$ :

$$m_{eff} = m \left( 1 + \left( \left. \frac{dy}{dx} \right|_{x=x_0} \right)^2 \right) = m \left( 1 + \left( \frac{\omega^2 x_0}{g} \right)^2 \right)$$

Thus, the angular frequency of oscillations is

$$\Omega = \sqrt{\frac{k_{eff}}{m_{eff}}} = \sqrt{\frac{m\omega^2(\lambda-2)}{m \left( 1 + \left( \frac{\omega^2 x_0}{g} \right)^2 \right)}} = \sqrt{\frac{\omega^2(\lambda-2)}{1 + \left( \frac{\omega^2(\lambda-1)a^{2\lambda}}{\lambda^2 b^2 g^2(\lambda-1)} \right)^{\frac{1}{\lambda-2}}}}$$

From the expression of  $\Omega$ , evidently, oscillations only exist when  $\lambda > 2$ .